Module 5: Predicate Logic
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- Pre-class quiz #6 is due Wednesday October 14th at 19:00
- Assigned reading for the quiz:
  - Epp, 4th or 5th edition: 6.1 up to (but not including) the part titled *Partition of Sets*.
  - Rosen, 7th edition: 2.1 except for the parts titled *The Size of a Set, Power Sets and Cartesian Products*. 2.2 up to but not including *Set Identities*.
  - Rosen, 6th edition: 2.1 except for the parts titled *Power Sets and Cartesian Products*. 2.2 up to but not including *Set Identities*. 
Module 5: Predicate Logic

- Pre-class quiz #7 is tentatively due Wednesday October 21st at 19:00.
  - Assigned reading for the quiz:
    - Epp, 4th or 5th edition: 3.2, 3.4
    - Epp, 3rd edition: 2.2, 2.4
    - Rosen, 6th edition: 1.3, 1.4
    - Rosen, 7th edition: 1.4, 1.5

Assignment #2 is due Wednesday October 14th at 19:00.
Module 5: Predicate Logic

Quiz 5 feedback:

- Very well done; only one question below 90%:
  - If D is the set of 8 bit signed integers, is
    \[ \forall x \in D, \forall y \in D, ((x > 0) \land (y > 0)) \rightarrow (x + y) > 0 \]
    true?
  - Answer: NO. For instance \(100 + 100 = -56\).

- As usual, we will discuss the open-ended question (do we need a quantifier for “exactly \(k\)”?) soon.
Module 5: Predicate Logic

- By the start of class, you should be able to
  - Evaluate the truth of predicates applied to particular values.
  - Show a predicate logic statement is true by enumerating examples, i.e. one (all) in the domain for an existential (universal) quantifier.
  - Show a predicate logic statement is false by enumerating counterexamples, i.e. all (one) in the domain for an existential (universal) quantifier.
  - Translate between statements in formal predicate logic notation and equivalent statements in closely matching informal language, i.e., informal statements with clear and explicitly stated quantifiers.
CPSC 121: the **BIG** questions:

- How can we convince ourselves that an algorithm does what it's supposed to do?
  - We need to *prove* that it works.
  - We have done a few proofs in the last week or so.
  - Many statements (that we need to prove) involve quantifiers.
Module 5: Predicate Logic

By the end of this module, you should be able to:

- Build statements about the relationships between properties of various objects using predicate logic. These may be
  - real-world like “every candidate got votes from at least two people in every province” or
  - computing related like “on the $i^{th}$ repetition of this algorithm, the variable $\text{min}$ contains the smallest element in the list between element 0 and element $i$”.
Module 5: Predicate Logic

- Module Summary
  - Predicates vs Propositions
  - Translating between English and Predicate Logic
  - There are exactly k such that ...
  - Statements with both $\forall$ and $\exists$
Module 5.1: Predicates vs Propositions

- What is predicate logic good for modeling?
  - Relationships among real-world objects.
  - Generalizations about patterns
  - Infinite domains
  - Generally, problems where the properties of the different concepts, or parts, depend on each other.
Module 5.1: Predicates vs Propositions

- Examples of predicate logic use in Computer Science:
  - Data structures: Every key stored in the left subtree of a node $N$ is smaller than the key stored at $N$. [CPSC 221]
  - Language definition: No path via references exists from any variable in scope to any memory location available for garbage collection... [CPSC 311 or 312]
  - Databases: the relational model is based on predicate logic. [CPSC 304]
  - Algorithms: in the worst case, every comparison sort requires at least $cn\log_2 n$ comparisons to sort $n$ values, for some constant $c > 0$. [CPSC 320]
Module 5.1: Predicates vs Propositions

- **Quantifiers scope:**
  - A quantifier applies to everything to its right, up to the closing parenthesis of the () pair that “contains” it.
  - **Example:**

\[
\forall x \in D, (\exists y \in E, Q(x, y) \rightarrow \forall z \in F, R(y, z)) \land P(x)
\]
Quantifiers scope:

- A quantifier applies to everything to its right, up to the closing parenthesis of the () pair that “contains” it.

Example:

\[ \forall x \in D, (\exists y \in E, Q(x, y) \rightarrow \forall z \in F, R(y, z)) \land P(x) \]
Quantifiers scope:

A quantifier applies to everything to its right, up to the closing parenthesis of the () pair that “contains” it.

Example:

\[ \forall x \in D, (\exists y \in E, Q(x, y) \rightarrow \forall z \in F, R(y, z)) \land P(x) \]
Negation scope:

What is being negated in the following statement?

\[ \neg \exists x \in \mathbb{Z}^+, \forall y \in \mathbb{Z}^+, \ x < y \land \text{Even}(y) \]?

a) The quantifier \( \exists \).
b) The quantified variable \( \exists x \in \mathbb{Z}^+ \).
c) The expression up to \( x < y \).
d) Everything to the right of the \( \neg \).
e) None of the above.
What is the truth value of the statement:

\[ \exists x \in \mathbb{Z}, x \cdot x = y \]

a) True because (for example) \( 5 \cdot 5 = 25 \)
b) True because every \( y \), we know \( y = \sqrt{y} \cdot \sqrt{y} \)
c) False, because of counterexamples like no integer multiplied by itself equals 3
d) It depends on \( y \), but given a value for \( y \), we could calculate a truth value.
e) None of the above.
Module 5.1: Predicates vs Propositions

What is the difference between a proposition and a predicate?

a) A predicate may contain unbound variables, but a proposition never does.

b) A predicate may contain one or more quantifiers, but a proposition never does.

c) A proposition's name is a lowercase letter, whereas a predicate's name is an uppercase letter.

d) They are the same thing, using different names.

e) None of the above.
Module 5.1: Predicates vs Propositions

- A predicate is a logic formula with unbound variables, such as
  \[ \text{Perfect Square}(y): \exists x \in \mathbb{Z}, x^2 = y \]

- Then
  - \text{PerfectSquare}(25) is \_\_\_\_
  - \text{PerfectSquare}(27) is \_\_\_\_
  - \( \exists y \in \mathbb{Z}, \text{PerfectSquare}(y) \) is \_\_\_\_
  - \( \forall y \in \mathbb{Z}, \text{PerfectSquare}(y) \) is \_\_\_\_
Module 5.1: Predicates vs Propositions

Which variables do we need values for in order to determine this formula's truth value?

\( \forall i \in \mathbb{Z}^+, (i \geq n) \leftrightarrow \neg \exists v \in \mathbb{Z}^+, \text{HasValue}(l, i, v) \)

a) i and v
b) l and n
c) n and v
d) i and n
e) None of these are correct.
Module 5: Predicate Logic

- Module Summary
  - Predicates vs Propositions
  - Translating between English and Predicate Logic
  - There are exactly k such that ...
  - Statements with both ∀ and ∃
Module 5.2: Translations

- Given the definitions:
  - $F$: the set of foods.
  - $E(x)$: Alice eats food $x$.
  - $g$: Alice grows.
  - $s$: Alice shrinks.

Worksheet problems #1 and #2
Module 5.2: Translations

- Given the definitions:
  - $F(x)$: $x$ is a fierce creature.
  - $L(x)$: $x$ is a lion
  - $C(x)$: $x$ drinks coffee
  - $D$: the set of all creatures.
  - $T(x,y)$: creature $x$ has “tasted” creature $y$.

Worksheet problems #3 and #4
Module 5.2: Translations

- Now let us look at how we restrict the domain associated with a quantifier using predicate logic.

Worksheet problems #5 to #10
Finally, let’s think about the problems caused by the fact that human languages are ambiguous:

Worksheet problem #11
Module 5: Predicate Logic

Module Summary

- Predicates vs Propositions
- Translating between English and Predicate Logic
- There are exactly k such that ...
- Statements with both ∀ and ∃
Module 5.3: There are exactly $k$ such that ...

- The statement
  \[ \exists x \in D, \ P(x) \]
  means
  there is at least one element with property $P$.

- Complete the following statement that means there are at least two elements with property $P$.
  \[ \exists x \in D, \ \exists y \in D, \]
Now complete the following statement that means there are at least three elements with property $P$.

$$\exists x \in D, \exists y \in D, \exists z \in D,$$

We can extend this to $n$ elements

- However the number of terms is about $n^2/2$.

Now let us turn out attention to at most.
Suppose I have a large urn with many tennis balls; at most one ball is coloured sky blue (the others are green).

- I pick one ball $B_1$ from the urn, it's sky blue.
- I put $B_1$ back in the urn, stir, and pick another ball $B_2$. It's also sky blue.

What can we say about $B_1$ and $B_2$?

- They are
Module 5.3: There are exactly k such that ...

- Complete the following statement that means there is at most one item with property $P$:
  $$\forall x \in D, \forall y \in D,$$

- Now complete the following statement that means there is at most two items with property $P$:
  $$\forall x \in D, \forall y \in D, \forall z \in D,$$

- There are *exactly* $k$ objects with property $P$ if there are *at least* $k$ objects with property $P$ and *at most* $k$ objects with property $P$. 
Module 5: Predicate Logic

- Module Summary
  - Predicates vs Propositions
  - Translating between English and Predicate Logic
  - There are exactly $k$ such that ...
  - Statements with both $\forall$ and $\exists$
Module 5.3: Statements with both $\forall$ and $\exists$

- Consider the following two statements:
  1. A rich person gave $10000 to every CPSC 121 student
  2. Every CPSC 121 student received $10000 from a rich person.

- Do they mean the same thing?
  a) Yes
  b) No
  c) It’s impossible to tell
Module 5.3: Statements with both $\forall$ and $\exists$

Consider

- $R$: the set of rich people
- $S$: the set of students
- $E_1$: A rich person gave $10000$ to every student.
- $E_2$: Every rich person gave $10000$ to a student.
- $E_3$: A student received $10000$ from every rich person.
- $E_4$: Every student received $10000$ from a rich person.
Module 5.3: Statements with both $\forall$ and $\exists$

Consider

- $P_1: \forall r \in R, \exists s \in S, \text{Gave}(r,s)$
- $P_2: \forall s \in S, \exists r \in R, \text{Gave}(r,s)$
- $P_3: \exists s \in S, \forall r \in R, \text{Gave}(r,s)$
- $P_4: \exists r \in R, \forall s \in S, \text{Gave}(r,s)$
Module 5.3: Statements with both $\forall$ and $\exists$

Which of the following lists of equivalences is correct?

a) $E_1 \equiv P_4, E_2 \equiv P_1, E_3 \equiv P_3, E_4 \equiv P_2$.

b) $E_1 \equiv P_2, E_2 \equiv P_3, E_3 \equiv P_1, E_4 \equiv P_4$.

c) $E_1 \equiv P_4, E_2 \equiv P_3, E_3 \equiv P_1, E_4 \equiv P_2$.

d) $E_1 \equiv P_2, E_2 \equiv P_1, E_3 \equiv P_3, E_4 \equiv P_4$.

e) None of the above.