Module 4: Propositional Logic Proofs
Module 4: Propositional Logic Proofs

- Pre-class quiz #5 is due Wednesday October 7th, at 19:00
- Assigned reading for the quiz:
  - Epp, 4th or 5th edition: 3.1, 3.3
  - Epp, 3rd edition: 2.1, 2.3
  - Rosen, 6th edition: 1.3, 1.4
  - Rosen, 7th edition: 1.4, 1.5
- Assignment #2 is due Wednesday October 14th at 19:00.
Module 4: Propositional Logic Proofs

- Pre-class quiz #6 is tentatively due Wednesday October 14\(^{th}\) at 19:00

- Assigned reading for the quiz:
  - Epp, 4\(^{th}\) or 5\(^{th}\) edition: 6.1 up to (but not including) the part titled \textit{Partition of Sets}.
  - Epp, 3\(^{rd}\) edition: 5.1 up to (but not including) the part titled \textit{Partition of Sets}.
  - Rosen, 7\(^{th}\) edition: 2.1 except for the parts titled \textit{The Size of a Set, Power Sets and Cartesian Products}. 2.2 up to but not including \textit{Set Identities}.
  - Rosen, 6\(^{th}\) edition: 2.1 except for the parts titled \textit{Power Sets and Cartesian Products}. 2.2 up to but not including \textit{Set Identities}.
Module 4: Propositional Logic Proofs

- By the start of this class you should be able to
  - Use truth tables to establish or refute the validity of a rule of inference.
  - Given a rule of inference and propositional logic statements that correspond to the rule's premises, apply the rule to infer a new statement implied by the original statements.
Module 4: Propositional Logic Proofs

Quiz 4 feedback:
- Very well done overall
- No question had an average below 90%.
- We will discuss the open-ended question soon.
Module 4: Propositional Logic Proofs

- CPSC 121: the BIG questions:
  - How can we convince ourselves that an algorithm does what it's supposed to do?
    - We need to prove that it works.
    - We have done a few proofs in the last week or so.
    - Now we will learn
      - How to decide if a proof is valid in a formal setting.
      - (soon) How to write proofs in English.
Module 4: Propositional Logic Proofs

- By the end of this module, you should be able to
  - Determine whether or not a propositional logic proof is valid, and explain why it is valid or invalid.
  - Explore the consequences of a set of propositional logic statements by application of equivalence and inference rules, especially in order to massage statements into a desired form.
  - Devise and attempt multiple different, appropriate strategies for proving a propositional logic statement follows from a list or premises.
Module 4: Propositional Logic Proofs

- Module outline
  - Proofs and their meaning.
  - Propositional Logic proofs.
  - Further exercises.
Module 4.1: Proofs and their meaning

What is a proof?

- A rigorous formal argument that demonstrates the truth of a proposition, given the truth of the proof’s premises.

In other words:

- A proof is used to convince other people (or yourself) of the truth of a conditional proposition.

- Every step must be well justified.

Writing a proof is a bit like writing a function:

- you do it step by step, and

- make sure that you understand how each step relates to the previous steps.
Module 4.1: Proofs and their meaning

- Things we might prove
  - We can build a combinational circuit matching any truth table.
  - We can build any combinational logic circuit using only 2-input XOR and ORNOT gates.
  - The maximum number of swaps we need to order \( n \) students is \( n(n-1)/2 \).
  - No general algorithm exists to sort \( n \) values using fewer than \( n \log_2 n \) comparisons.
  - There are problems that no algorithm can solve.
What does this argument mean?:

Premise 1

...

Premise n

.: Conclusion

a) Premise1 ^ ... ^ Premise n ^ Conclusion

b) Premise1 v ... v Premise n v Conclusion

c) (Premise1 ^ ... ^ Premise n) → Conclusion

d) (Premise1 ^ ... ^ Premise n) ↔ Conclusion

e) None of the above.
Module 4.1: Proofs and their meaning

Suppose that you proved this:

Premise 1

... 

Premise n

\[ \therefore \text{ Conclusion} \]

Does it mean:

a) Premises 1 to n are true
b) Conclusion is true
c) Premises 1 to n are not a contradiction
d) Conclusion isn't a contradiction
e) None of the above.
Module 4.1: Proofs and their meaning

- Let's consider an invalid rule:
  \[ p \to q \]
  \[ q \]
  \[ \therefore p \]

- What can we say about the truth value of \( p \)?
  a) \( p \) is true
  b) \( p \) is false
  c) \( p \) might be either true or false
  d) \( p \) can be neither true nor false
Module 4: Propositional Logic Proofs

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Module 4.2: Propositional Logic proofs

- A propositional logic proof is a sequence of propositions, where each proposition is one of:
  - A premise
  - The result of applying a logical equivalence or a rule of inference to one or more earlier propositions.
- and whose last proposition is the conclusion.
- These are good starting point, because they are simpler than the more free-form proofs we will discuss later
  - Only a limited number of choices at each step.
## Module 4.2: Rules of inference

<table>
<thead>
<tr>
<th>Rule</th>
<th>Premise 1</th>
<th>Premise 2</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Modus Ponens</strong>  &lt;br&gt; [M. PON]</td>
<td>( p \rightarrow q )</td>
<td>( p )</td>
<td>( \therefore q )</td>
</tr>
<tr>
<td><strong>Modus Tollens</strong>  &lt;br&gt; [M. TOL]</td>
<td>( p \rightarrow q )</td>
<td>( \neg q )</td>
<td>( \therefore \neg p )</td>
</tr>
<tr>
<td><strong>Generalization</strong>  &lt;br&gt; [GEN]</td>
<td>( p )</td>
<td>( \therefore p \lor q )</td>
<td>( p \lor q \rightarrow q \rightarrow p )</td>
</tr>
<tr>
<td><strong>Specialization</strong>  &lt;br&gt; [SPEC]</td>
<td>( p \land q )</td>
<td>( p )</td>
<td>( \therefore q )</td>
</tr>
<tr>
<td><strong>Conjunction</strong>  &lt;br&gt; [CONJ]</td>
<td>( p \land q \rightarrow p \rightarrow q \rightarrow p \rightarrow r )</td>
<td>( q \rightarrow r \rightarrow p \rightarrow r )</td>
<td>( \therefore p \land q \rightarrow p \rightarrow r )</td>
</tr>
<tr>
<td><strong>Elimination</strong>  &lt;br&gt; [ELIM]</td>
<td>( p \lor q )</td>
<td>( \neg p )</td>
<td>( \therefore q )</td>
</tr>
<tr>
<td><strong>Proof by cases</strong>  &lt;br&gt; [CASE]</td>
<td>( p \rightarrow r )</td>
<td>( q \rightarrow r \rightarrow (p \lor q) \rightarrow r )</td>
<td>( \therefore (p \lor q) \rightarrow r )</td>
</tr>
<tr>
<td><strong>Transitivity</strong>  &lt;br&gt; [TRANS]</td>
<td>( p \rightarrow q )</td>
<td>( q \rightarrow r \rightarrow p \rightarrow r )</td>
<td></td>
</tr>
<tr>
<td><strong>Contradiction</strong>  &lt;br&gt; [CONT]</td>
<td>( p \rightarrow \bot )</td>
<td>( \therefore \neg p )</td>
<td></td>
</tr>
</tbody>
</table>
Module 4.2: The fire-trolls problem

- Fire-trolls problem from pre-class quiz #4
  - Critique the following argument, paraphrased from an article by Julian Baggini on logical fallacies.
    - Premise 1: If dragons are too scaly to portray dragons then trolls must be too smelly to play trolls, and vice versa.
    - Premise 2: And yet, if the fire-trolls are correct, dragons are too scaly to portray dragons and yet trolls are not too smelly to play trolls.
    - Conclusion: Therefore, the fire-trolls are incorrect, and dragons are not too scaly to portray dragons.
  - Note: fire-trolls are trolls portraying dragons in mystical theater.
Module 4.2: The fire-trolls problem

- Fire-trolls: which definitions should we use?
  
a) \(d = \text{dragons}, \ t = \text{trolls}, \ sc = \text{scaly}, \ sm = \text{smelly}, \ ft = \text{fire-trolls}, \ c = \text{correct}\)
  
b) \(d = \text{dragons are too scaly, } t = \text{trolls are too smelly, } pd = \text{dragons portray dragons, } pt = \text{trolls portray trolls, } o = \text{fire-trolls are correct}\)
  
c) \(d = \text{dragons are too scaly to portray dragons, } t = \text{trolls are too smelly to portray trolls, } o = \text{fire-trolls are correct}\)
  
d) None of these, but another set of definitions works well.
  
e) None of these, and this problem cannot be modeled well with propositional logic.
Module 4.2: The fire-trolls problem

- Fire-trolls: do the two premises contradict each other (that is, is $p_1 \land p_2 \equiv F$)?
  
  a) Yes
  
  b) No
  
  c) Not enough information to tell
Module 4.2: The fire-trolls problem

- What can we prove?
  - We can prove that the fire-trolls are wrong.
  - We can **not** prove that dragons are not too scaly to portray dragons.

  - What other scenario is consistent with the premises?
Module 4.2: Propositional Logic proofs

- Proof strategies
  - Look at the information you have
    - Is there irrelevant information you can ignore?
    - Is there critical information you should focus on?
  - Work backwards from the end
    - Especially if you have made some progress but are missing a step or two.
  - Don't be afraid of inferring new propositions, even if you are not quite sure whether or not they will help you get to the conclusion you want.
Module 4.2: Propositional Logic proofs

- Proof strategies (continued):
  - If you are not sure of the conclusion, alternate between
    - trying to find an example that shows the statement is false, using the place where your proof failed to help you design the counterexample.
    - trying to prove it, using your failed counterexample to help you write the proof.
Example: prove that the following argument is valid:

\[
\begin{align*}
p \\
p \rightarrow r \\
p \rightarrow \neg s \\
p \rightarrow (q \lor \neg r) \\
\neg q \lor s \\
\therefore s
\end{align*}
\]
Why can we not just use truth tables to prove propositional logic theorems?

a) No reason; truth tables are enough.

b) Truth tables scale poorly to large problems.

c) Rules of inference can prove theorems that cannot be proven with truth tables.

d) Truth tables require insight to use, while rules of inference can be applied mechanically.
Why not use logical equivalences to prove that the conclusions follow from the premises?

a) No reason; logical equivalences are enough.

b) Logical equivalences scale poorly to large problems.

c) Rules of inference can prove theorems that cannot be proven with logical equivalences.

d) Logical equivalences require insight to use, while rules of inference can be applied mechanically.
Module 4.2: Propositional Logic proofs

- One last question:
  - Consider the following:
    - Patrice is rich
    - If Patrice is rich then he will pay your tuition
    - \(\therefore\) Patrice will pay your tuition.
  - Is this argument valid?
  - Should you pay your tuition, or should you assume that Patrice will pay it for you? Why?
Module 4: Propositional Logic Proofs

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  - Further exercises.
Module 4.3: Further exercises

- Prove that the following argument is valid:

  \[
  \begin{align*}
  &p \rightarrow q \\
  &q \rightarrow (r \land s) \\
  &\sim r \lor (\sim t \lor u) \\
  &p \land t \\
  \hline
  \therefore &u
  \end{align*}
  \]

- Given the following, what is everything you can prove?

  \[
  \begin{align*}
  &p \rightarrow q \\
  &p \lor \sim q \lor r \\
  &(r \land \sim p) \lor s \lor \sim p \\
  &\sim r
  \end{align*}
  \]
Further exercises

Hercule Poirot has been asked by Lord Rumpd Dalton to find out who closed the lid of his piano after dumping the cat inside. Poirot interrogates two of the servants, Meece Pink and Jhyl Klone. One and only one of them put the cat in the piano. Plus, one always lies and one never lies.

- Jhyl Klone: I did not put the cat in the piano. Ayul Parn gave me less than $60 to help her study.
- Meece Pink: Jhyl Klone did it. Ayul Parn paid him $50 to help her study.

Who put the cat in the piano?