CPSC 320: Intermediate Algorithm Design 2019/20 Term 2

Tutorial 11

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1 4-coloring a graph is NP-complete

Graph coloring is discussed in the textbook (Kleinberg-Tardos Section 8.7, Erickson Section 12.10). Given a graph $G = (V, E)$, a k-coloring of G is a function f mapping the vertices V to the integers $\{1, ..., k\}$, such that $f(u) \neq f(v)$ for every edge $uv \in E$.

The **3-Coloring** problem is "Given a graph G, decide if G has a 3-coloring."

The **4-Coloring** problem is "Given a graph G, decide if G has a 4-coloring."

1. Prove that the 4-Coloring problem is in NP.

2. Intuitively, the 4-Coloring problem should be easier than the 3-Coloring problem because you have more colors to work with.

   Nevertheless, we will give a reduction from the 3-Coloring problem to the 4-Coloring problem. This means that 4-Coloring is "at least as hard" as 3-Coloring, which contradicts our intuition.

   Key technical step. Given a graph G (for which we do not know whether it is 3-Colorable), define a new graph G' (which may or may not be 4-Colorable), satisfying the following key property:

   $$\textit{G is 3-Colorable if and only if G' is 4-Colorable.}$$

   Hint. 4-Coloring seems easier because we have the fourth color to use. However, we can construct G' (based on G) so that the fourth color becomes useless.

3. Given a 3-Coloring of G, explain how to produce a 4-Coloring of G'.

4. Given a 4-Coloring of G', explain how to produce a 3-Coloring of G.

5. Show that 3-Coloring $\leq_P$ 4-Coloring.

6. The 3-Coloring problem is known to be NP-complete. (See Kleinberg-Tardos Section 8.7 and Erickson Section 12.10.) Prove that the 4-Coloring problem is NP-complete, using Karp's recipe (described in Lecture 33).

2 Bonus: Planar graphs

A graph is planar if it can be drawn in the plane so that edges don't cross.

The **Planar-3-Coloring** problem is "Given a planar graph G, decide if G has a 3-coloring."

The **Planar-4-Coloring** problem is "Given a planar graph G, decide if G has a 4-coloring."

It is known that Planar-3-Coloring is NP-complete.

7. **KEENER KWESTION.** Does our reduction above prove that Planar-3-Coloring $\leq_P$ Planar-4-Coloring?

8. **KEENER KWESTION.** Prove that Planar-4-Coloring is in P.