# CPSC 424
# Differential Geometry of Curves

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## Syllabus

***Curves in 2D and 3D***

- Implicit vs. Explicit vs. Parametric curves
- Parametric Curves: Basis functions and Splines
- Splines: Bezier and Hermite
- Other bases
- **Elementary differential geometry**
- Subdivision Curves
- 2D Implicits: Analytical and Neural

***Surfaces***

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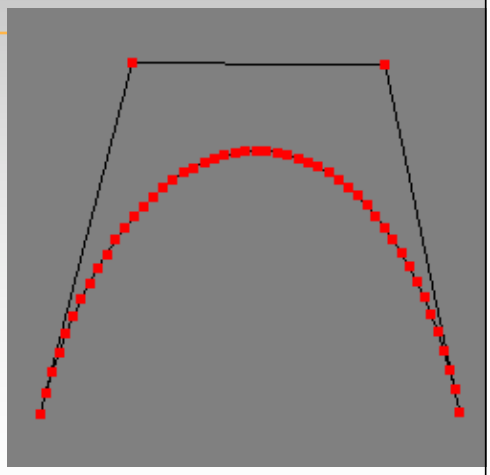
# Motivation



***Parameterization:***

- Curve representations we have looked at have fixed parameterization
  - *Bézier*
  - *B-Spline*
  - *Rational*
- Indicates fixed speed for point traveling along the curve

***Want:***

- Change speed without affecting shape

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# Motivation

***Continuity between curves***

- So far included parameterizations
- Example:
  - *Bézier continuity constraints depend on relative interval lengths*
  - *B-Spline continuity depends on interval lengths in knot sequence*

***Want:***

- Geometric concept for continuity that ignores parameterization

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# Motivation

***Differential Geometry helps to:***

- Understand parameterizations
- Understand impact of parameterizations on derivatives
  - *Tangent vector, curvature, …*
- Change parameterization as required
- Derive geometric definition of continuity

***For our purposes:***

- Look at 3D curves
- 2D as special case

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# Regularity

***Definition:***

- Differentiable parametric curve $F(t):[a,b]\mapsto \mathbb{R}^3$ is called regular if

$$F'(t)\neq 0, \forall t\in[a,b]$$

- (I.e. if the tangent vector is not 0 anywhere)

***Note:***

- Bézier curves not necessarily regular…

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# Equivalence / Reparameterization

***Definition***

- Two regular curves

$$F(t):[a,b]\mapsto \mathbb{R}^3 \qquad G(t):[c,d]\mapsto \mathbb{R}^3$$

are geometrically equivalent $F \cong G$ if there is a strictly monotonic, differentiable function

$$\varphi(t):[a,b]\mapsto[c,d]$$

with

$$F(t)=G(\varphi(t))$$

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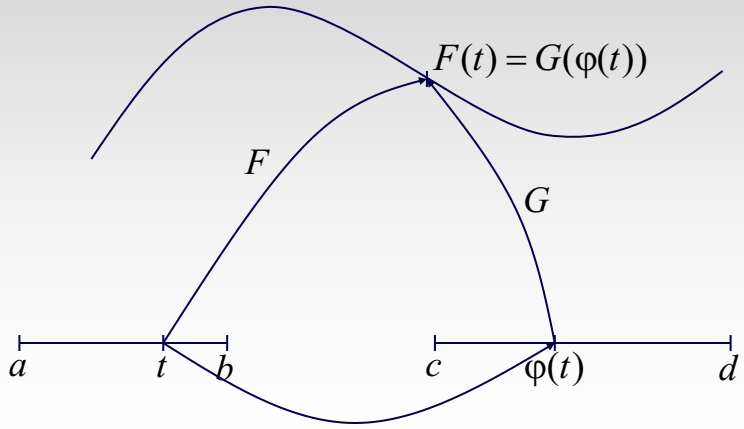
# Equivalence / Reparameterization

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## Question

*Are the curves*

F(t) = (t,t)  t in [0,1] and G(t) = (2t,2t) t in [0,0.5]

*geometrically equivalent?*

- A. Yes
- B. No
- C. Not enough information

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## Arc Length

***Definition***

- Arc length of regular curve $F(t):[a,b]\mapsto \mathbb{R}^3$ given as

$$s(t) := \int_a^t \| F'(\tilde{t}) \| \, d\tilde{t}$$

***Parameterization by arc length***

$$G(s) \text{ with } G(s(t)) = F(t)$$

- Note: this is a **canonical** representation for any curve
- Point is traveling along $G$ with constant speed 1

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## Arc Length Example

F(t) = (t,t)  t in [0,1]

F'(t)= ?  G(s) = ?

$$s(t) := \int_a^t \| F'(\tilde{t}) \| \, d\tilde{t}$$

F(t) = (sin(t),cos(t))  t in [0,π]

F'(t)= ?  G(s) = ?

$$G(s) \text{ with } G(s(t)) = F(t)$$

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## Curvature

***Definition***

- Let $G$ be a curve parameterized by arc length
- We introduce the following terms:
  - *Unit tangent* $T(s) := G'(s)$
  - *Curvature vector* $K(s) := G''(s)$
  - *Curvature* $\kappa(s) := \| K(s) \|$
  - *Principal normal* $N(s) := K(s) / \kappa(s)$
  - *Binormal* $B(s) := T(s) \times N(s)$

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# General Setting

- To translate into general parameterization use derivation rules
  - *e.g. Unit tangent* $T(t) := \frac{dG}{ds} = \frac{dF}{dt}\frac{dt}{ds} = \frac{F'}{\|F'\|}$

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# Curvature

- What is Curvature?

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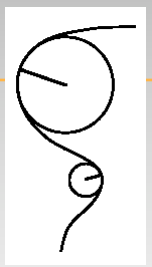
## Curvature

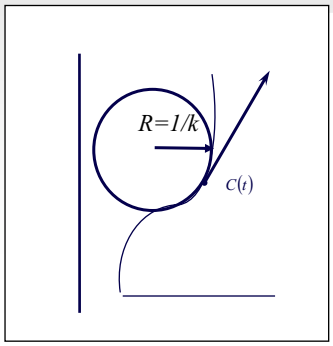
- General parameterization (2D)

$$k(t) = \frac{x'(t)y''(t) - y'(t)x''(t)}{\left(x'(t)^2 + y'(t)^2\right)^{3/2}}$$



***Corresponds to radius of osculating circle* R=1/k**

***Measures curve bending***

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## Frenet Frame

***Theorem:***

- Curvature vector and tangent vector are perpendicular:

$$K(s) \perp T(s)$$

***Note:***

- Therefore, T, N, and B form an orthonormal coordinate frame
- This is called the Frenet Frame

**https://www.desmos.com/3d/ei5yo6m9al**

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# Torsion

***With a similar argument to curvature we get***

$$B'(s) = \tau(s) \cdot N(s)$$

***Note:***

- $B'$ is the torsion vector
- $\tau$ is the torsion, and indicates how much the curve twists out of the plane ($\tau$=0 means perfectly planar)

**https://www.desmos.com/3d/fd2zkrdg8w**

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# Fundamental Theorem of Curves

***Theorem:***

- For given functions $\kappa(s)$, $\tau(s)$ there exists exactly one (except for rotations and translations) unique curve that is parameterized by arc length and has curvature $\kappa(s)$, and torsion $\tau(s)$

***Proof:***

- Quite complex, see for example
  - *do Carmo*
    *Differential Geometry of Curves and Surfaces*

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# Geometric Continuity

## *Definition:*

- Two curves

$$F_1(t):[a,b]\mapsto \mathbb{R}^3, F_2(t):[b,c]\mapsto \mathbb{R}^3$$

are $G^k$-continuous (geometrically continuous of degree k), if there are reparameterizations

$$G_1(t)\cong F_1(t) \text{ and } G_2(t)\cong F_2(t)$$

that are $C^k$ continuous, i.e.:

$$G_1^l(t)=G_2^l(t), l=0\ldots k$$

at shared parameter interval endpoint

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# Geometric Continuity

## *Note:*

- In practice, if two curves are $G^k$ continuous, then there always is a reparameterization for $F_2$ that is $C^k$ continuous to $F_1$
  - *I.e. only one of the two curves needs to be reparameterized*
- If we reparameterize both curves, we can normalize the tangent vector at the transition point to unit length
  - *Local arc length parameterization*

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