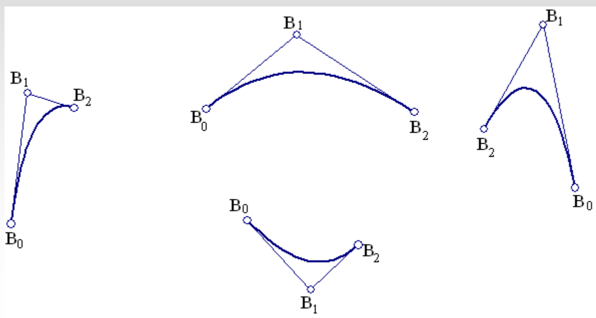
CPSC 424
Hermite Curves & Splines



Syllabus

Curves in 2D and 3D

- Implicit vs. Explicit vs. Parametric curves
- Parametric Curves: Basis functions and Splines
- **Splines:** Bezier and **Hermite**
- Elementary differential geometry
- Other bases
- Subdivision Curves
- 2D Implicits: Analytical and Neural

Surfaces

Back to Bézier Curves

Advantage of Bézier curves:

- Control points & control polygon have clear geometric meaning and are intuitive to use

Bezier Curves: Are we done?

Problem: how to describe long curves?

Using high-order Bezier curve = approximation quality diminishes

Reality: Only Quadratic & Cubic Bezier Curves are "usable"

Splines

Concept:

- Provide local control by piecing together multiple (polynomial) curves in smooth fashion
- This is called Spline
- Like with polynomial curves there are multiple representations:
  - ***Bezier Spline***
  - ***Hermite Spline***
  - *B-Spline & co…*

Continuity

Tangent Vector of Parametric Curve:

- Given by its derivative (here: 3D):

$$T(t) = F'(t) = \begin{pmatrix} dx/dt \\ dy/dt \\ dz/dt \end{pmatrix}$$

- Example (2D explicit curve):

$$F(t) := \begin{pmatrix} t \\ t^2 \end{pmatrix} \qquad T(t) := F'(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix}$$

Continuity

Note:

- This is a tangent vector, not a tangent direction
- This vector has a length
  - *Represents speed of an object moving along the curve*

Continuity

Def:

- A curve $F(t)$ is called C^k-continuous if its k^{th} derivative $F^{(k)}(t)$ exists (i.e. is continuous) everywhere

Note:

- Polynomial curves are infinitely continuous

Def:

- Two curve segments $F(t)$ defined over $[t,t_0]$ and $G(t)$ defined over $[t_0,t']$ are called C^k-continuous at t_0 if their first k derivatives match at t_0
  - *Definition extends to cases with "shifted" parameter intervals $F(t)$ $[t,t_0]$ and $G(t)$ $[t_1,t']$ are called C^k-continuous if at if first k derivatives of F(t) at t_0 match first k derivatives of G(t) at t_1*

Continuity

Examples:

- C^0-continuous: $F(t_0) = G(t_0)$
  - *I.e. the curves meet in one point for the same parameter value*
- C^1-continuous: $F(t_0) = G(t_0)$ and $F'(t_0) = G'(t_0)$
  - *They meet and have the same tangent vector*
  - *Note: both direction and length of vector are important!*
- C^2-continuous: in addition $F''(t_0) = G''(t_0)$
  - *Curvatures match as well*
- …

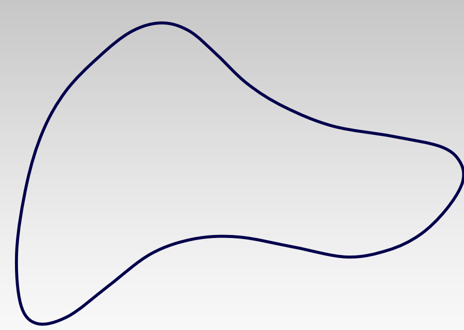
Geometric Continuity (informal/ more later)

Examples:

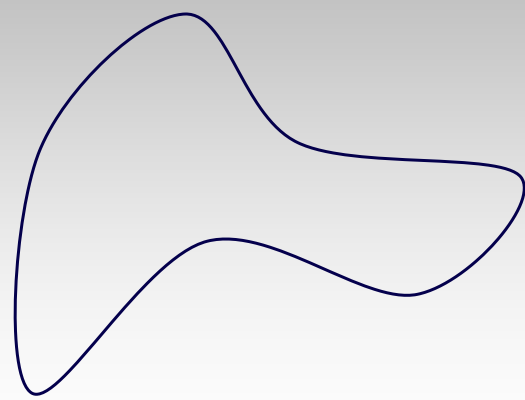
- G^0-continuous: $F(t_0) = G(t_0)$
  - *I.e. the curves meet in one point for the same parameter value*
- G^1-continuous: $F(t_0) = G(t_0)$ and $F'(t_0) = \mathbf{c}\, G'(t_0)$
  - $\mathbf{c}$ – *a constant (same for all dimensions)*
  - *They meet and have the same tangent* ***direction***

G1/C1

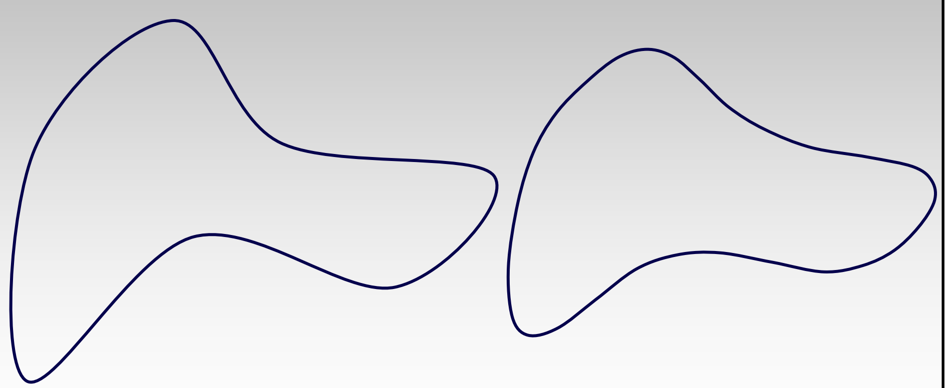
C1 and C2 continuity



C1 and C2 continuity



C1 and C2 continuity



Bezier Continuity

- ***C^0 :*** share end control points $b_m = b'_0$
- ***C^1 :*** $b_m - b_{m-1} = b'_1 - b'_0$
- ***G^1 :*** $b_m - b_{m-1}$ collinear to $b'_1 - b'_0$

Examples (on board)

Continuity Between Bézier Curves

Remarks:

- For C^k – continuity, the first (k+1) control points of G(t) are fixed
  - *Given as affine combinations of the control points of F(t)*
  - *NOT: convex combinations*
    - There will be coefficients >1 and <0
    - Extrapolation rather than interpolation

Degree Elevation

Motivation:

- Sometimes necessary to view curve of degree m as curve of degree m+1 or higher
  - *E.g. enforce continuity*
- Control points of degree m Bézier curve can be geometrically converted into degree (m+1) control points for **same** curve

Degree Elevation

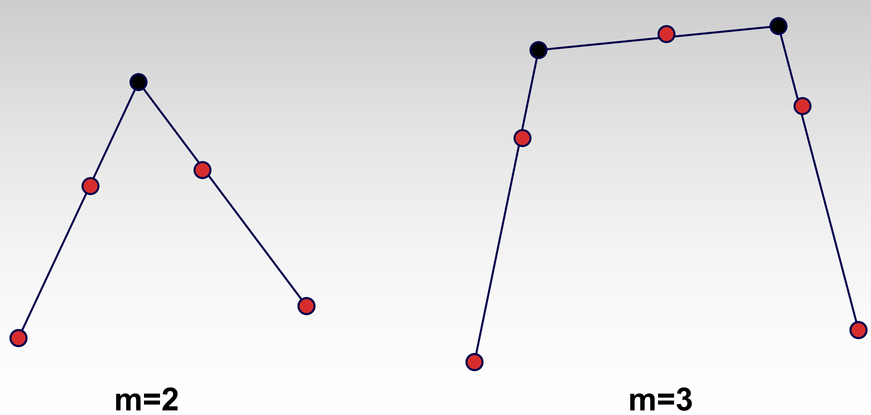
Replace degree m polynomial (m+1 control points) with degree m+1 polynomial (m+2 control points):

- New control points $\mathbf{b}'_i$:

$$\mathbf{b}'_i = \frac{1}{m+1}\left[i \cdot \mathbf{b}_{i-1} + (m+1-i)\mathbf{b}_i\right]$$

Degree Elevation

Examples:



Discussion

Degree Elevation:

- Provides additional control
  - *one more control point to manipulate*

But:

- Control is NOT local
  - *Moving one control point changes curve over whole parameter interval*
  - *Follows from de Casteljau algorithm*
- May want local control:
  - *Every control point affects small region of curve*

More Advanced…

- Getting continuity (C1/C2) “under the hood” (2026)

https://graphics.cs.utah.edu/research/projects/continuity-enhancement/

Other Option: Hermite Curves

Building block for C1 splines

Other Option: Hermite Cubic Basis

Geometrically-oriented coefficients

- 2 positions + 2 tangents

Require $F(0)=P_0,\ F(1) = P_1,\ F'(0)=T_0,\ F'(1)=T_1$

Define basis function per requirement

$$F(t) = P_0 h_{00}(t) + P_1 h_{01}(t) + T_0 h_{10}(t) + T_1 h_{11}(t)$$

Hermite Basis Functions

$$F(t) = P_0 h_{00}(t) + P_1 h_{01}(t) + T_0 h_{10}(t) + T_1 h_{11}(t)$$

To enforce** $C(0)=P_0$, $C(1) = P_1$, $C'(0)=T_0$, $C'(1)=T_1$ **basis should satisfy

$$h_{ij}(t): i, j = 0,1, t \in [0,1]$$

curve	$F(0)$	$F(1)$	$F'(0)$	$F'(1)$
$h_{00}(t)$	1	0	0	0
$h_{01}(t)$	0	1	0	0
$h_{10}(t)$	0	0	1	0
$h_{11}(t)$	0	0	0	1

Hermite Cubic Basis

Can satisfy with cubic polynomials as basis

$$h_{ij}(t) = a_3 t^3 + a_2 t^2 + a_1 t + a_0$$

Obtain - solve 4 linear equations in 4 unknowns for each basis function

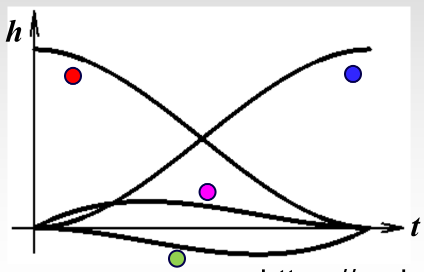
$$h_{ij}(t): i, j = 0,1, t \in [0,1]$$

curve	$F(0)$	$F(1)$	$F'(0)$	$F'(1)$
$h_{00}(t)$	1	0	0	0
$h_{01}(t)$	0	1	0	0
$h_{10}(t)$	0	0	1	0
$h_{11}(t)$	0	0	0	1

Hermite Cubic Basis

Four polynomials that satisfy the conditions

$$h_{00}(t) = t^2(2t-3)+1 \qquad h_{01}(t) = -t^2(2t-3)$$
$$h_{10}(t) = t(t-1)^2 \qquad h_{11}(t) = t^2(t-1)$$

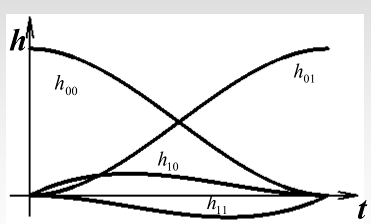


https://codepen.io/liorda/pen/KrvBwr

Hermite Cubic Basis

Four polynomials that satisfy the conditions

$$h_{00}(t) = t^2(2t-3)+1 \qquad h_{01}(t) = -t^2(2t-3)$$
$$h_{10}(t) = t(t-1)^2 \qquad h_{11}(t) = t^2(t-1)$$



https://codepen.io/liorda/pen/KrvBwr

Hermite Curves

Challenges:

- Setting magnitude of tangents
- Continuity across longer splines

Demo: https://codepen.io/liorda/pen/KMvBwM

Are Bezier and Hermite Related?

Are Bezier and Hermite Related?

Yes…..

How?