CPSC 424 - Tutorial 2

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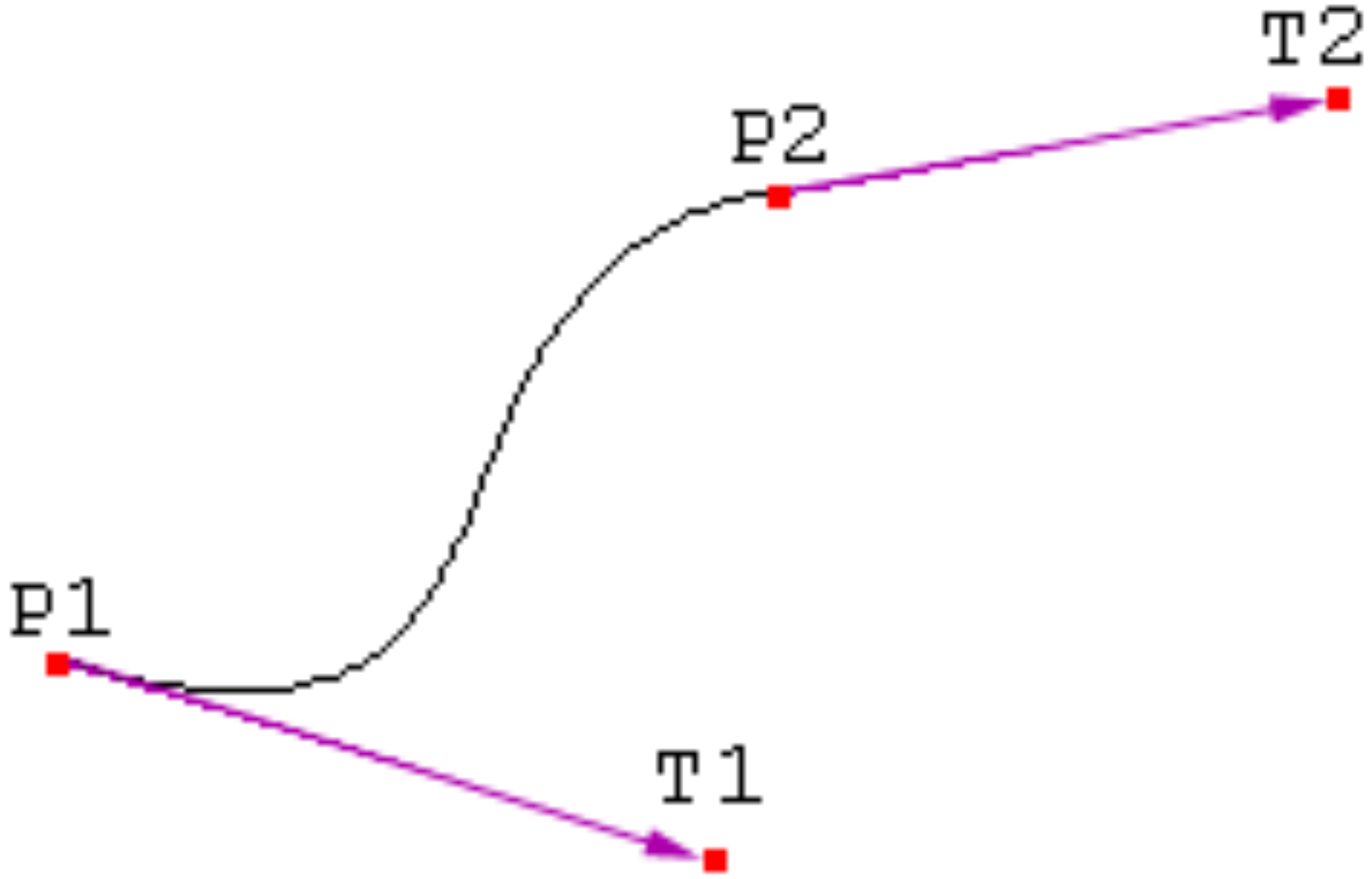
Hermite Curves



https://www.cubic.org/docs/hermite.htm

Curve formed from 2 positions P_0, P_1 and their tangents T_0, T_1

The curve is formulated as
$F(t) = P_0 h_{00}(t) + P_1 h_{01}(t) + T_0 h_{10}(t) + T_1 h_{11}(t)$

Such that
$F(0) = P_0, F(1) = P_1, F'(0) = T_0, F'(1) = T_1$

curve	***F*(0)**	***F*(1)**	***F'*(0)**	***F'*(1)**
$h_{00}(t)$	**1**	**0**	**0**	**0**
$h_{01}(t)$	**0**	**1**	**0**	**0**
$h_{10}(t)$	**0**	**0**	**1**	**0**
$h_{11}(t)$	**0**	**0**	**0**	**1**

Hermite Curves

Derivation

Since Hermite curves are cubic polynomials, we represent them as:

$$F(t) = at^3 + bt^2 + ct + d$$

And its derivative as:

$$F'(t) = 3at^2 + 2bt + c$$

Hermite Curves
Derivation

$F(t) = at^3 + bt^2 + ct + d$

$F'(t) = 3at^2 + 2bt + c$

Recall that for our curves we constraint their position and tangent at $t = 0, t = 1$

$F(0) = a(0) + b(0) + c(0) + d = d$

$F(1) = a + b + c + d$

$F'(0) = c$

$F'(1) = 3a + 2b + c$

Hermite Curves

Derivation

$$F(0) = a(0) + b(0) + c(0) + d = d$$

$$F(1) = a + b + c + d$$

$$F'(0) = c$$

$$F'(1) = 3a + 2b + c$$

And as a matrix:

$$\begin{bmatrix} P(0) \\ P(1) \\ T(0) \\ T(1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Hermite Curves

Derivation

Performing some matrix operations we can set it equal to a, b, c, d

$$\begin{bmatrix} P(0) \\ P(1) \\ T(0) \\ T(1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} P(0) \\ P(1) \\ T(0) \\ T(1) \end{bmatrix}$$

Hermite Curves

Derivation

Performing some matrix operations we can set it equal to a, b, c, d

And after taking the inverse of the matrix, we get the Hermite matrix, H_M

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P(0) \\ P(1) \\ T(0) \\ T(1) \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = H_M \begin{bmatrix} P(0) \\ P(1) \\ T(0) \\ T(1) \end{bmatrix}$$

Hermite Curves

Derivation

Recall that F(t) can also be represented as a matrix multiplication:

$$F(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$F(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P(0) \\ P(1) \\ T(0) \\ T(1) \end{bmatrix}$$

Hermite Curves

Derivation

After multiplying out and grouping by P_0, P_1, T_0, T_1 we get our blending polynomials $h(t)$

$$F(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P(0) \\ P(1) \\ T(0) \\ T(1) \end{bmatrix}$$

$$F(t) = P_0 h_{00}(t) + P_1 h_{01}(t) + T_0 h_{10}(t) + T_1 h_{11}(t)$$

$$\begin{aligned} h_{00}(t) &= 2t^3 - 3t^2 + 0t + 1 \\ h_{01}(t) &= -2t^3 + 3t^2 + 0t + 0 \\ h_{10}(t) &= t^3 - 2t^2 + t + 0 \\ h_{11}(t) &= t^3 - t^2 + 0t + 0. \end{aligned}$$

Bezier Curves

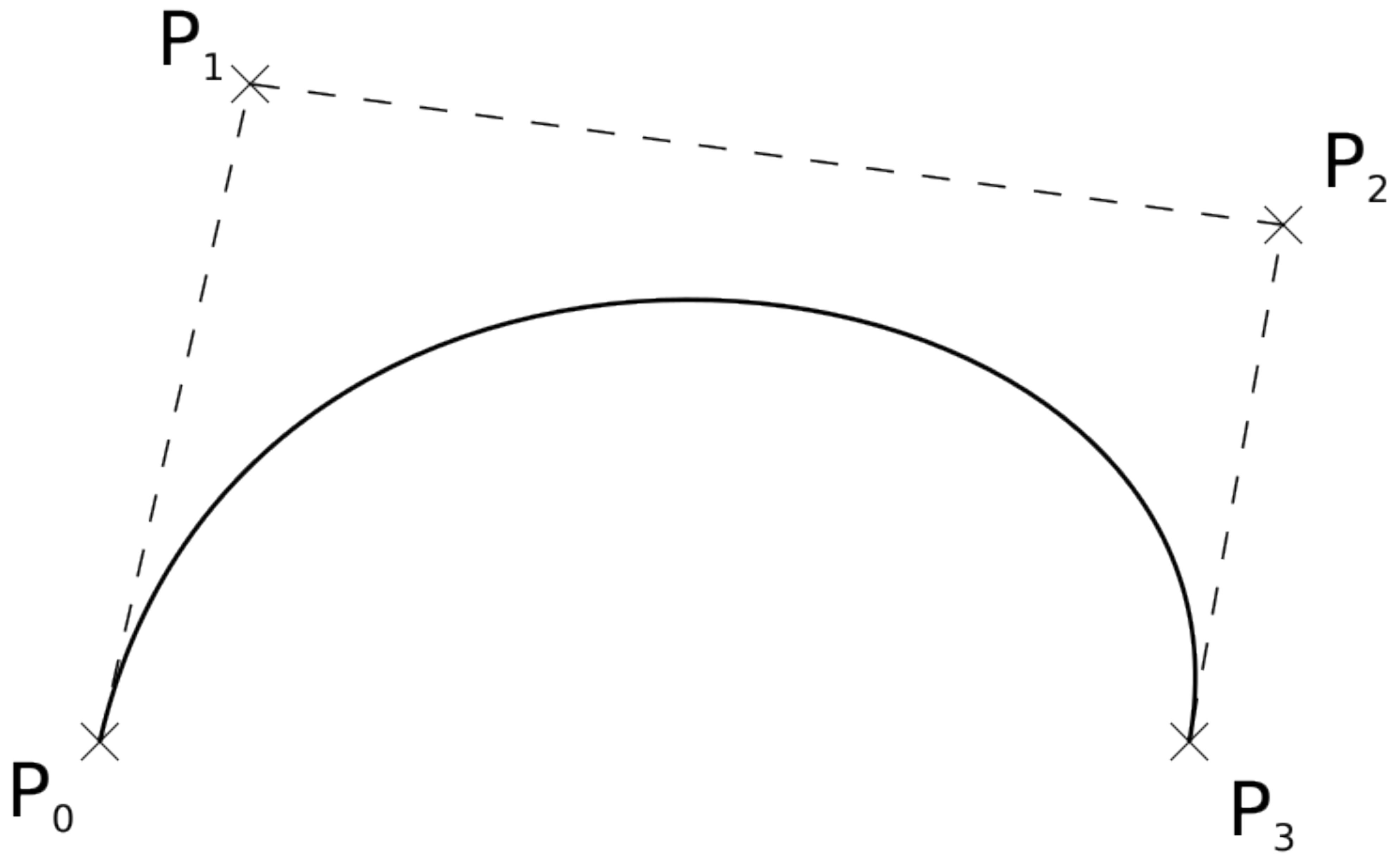
Overview

Unlike the previous curves we saw, such as Lagrange and Hermite, Bezier curves are a non-interpolating scheme (curve doesn't go through all the points).

Given m+1 control points, Bezier curves use Bernstein polynomials as their basis functions:

$$B_i^m(t) = \binom{m}{i} t^i (1-t)^{m-i}, \quad i = 0, \ldots, m; \quad 0 \le t \le 1$$

$$\binom{m}{i} = \frac{m!}{(m-i)!\, i!}$$



Wikipedia

Bezier Curves

Bernstein Polynomials

Exercise 1. Compute $B_0^1(t)$ and $B_1^1(t)$.

Exercise 2. Compute $B_0^{64}(t)$ and $B_{64}^{64}(t)$.

Exercise 3. Compute $B_2^5(t)$.

$$B_i^m(t) = \binom{m}{i} t^i (1-t)^{m-i}, \quad i = 0,\ldots,m; \quad 0 \le t \le 1$$

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Bezier Curves

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$$\begin{aligned} B_0^1(t) &= \binom{1}{0} t^0 (1-t)^{1-0} \\ &= 1 - t \\ B_1^1(t) &= \binom{1}{1} t^1 (1-t)^{1-1} \\ &= t \end{aligned}$$

Bezier Curves

Bernstein Polynomials

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$$\binom{m}{i} = \frac{m!}{(m-i)!\, i!}$$

$$B_0^{64}(t) = \binom{64}{0} t^0 (1-t)^{64-0}$$

$$= (1-t)^{64}$$

$$B_{64}^{64}(t) = \binom{64}{64} t^{64} (1-t)^{64-64}$$

$$= t^{64}$$

Bezier Curves

Bernstein Polynomials

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$$\binom{m}{i} = \frac{m!}{(m-i)!\,i!}$$

$$\begin{aligned} B_2^5(t) &= \binom{5}{2} t^2 (1-t)^{5-2} \\ &= \frac{5!}{3!\cdot 2!} t^2 (1-t)^3 \\ &= 10t^2(1-t)^3 \end{aligned}$$

Bezier Curves

Bernstein Polynomials

For this next exercise, we use the Binomial Theorem:

$$(x+y)^m = \sum_{k=0}^{m} \binom{m}{k} x^{m-k} y^k .$$

Exercise 4. Show that $\sum_{i=0}^{m} B_i^m(t) = 1$

$$B_i^m(t) = \binom{m}{i} t^i (1-t)^{m-i}, \quad i = 0, \ldots, m; \quad 0 \le t \le 1$$

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Bezier Curves

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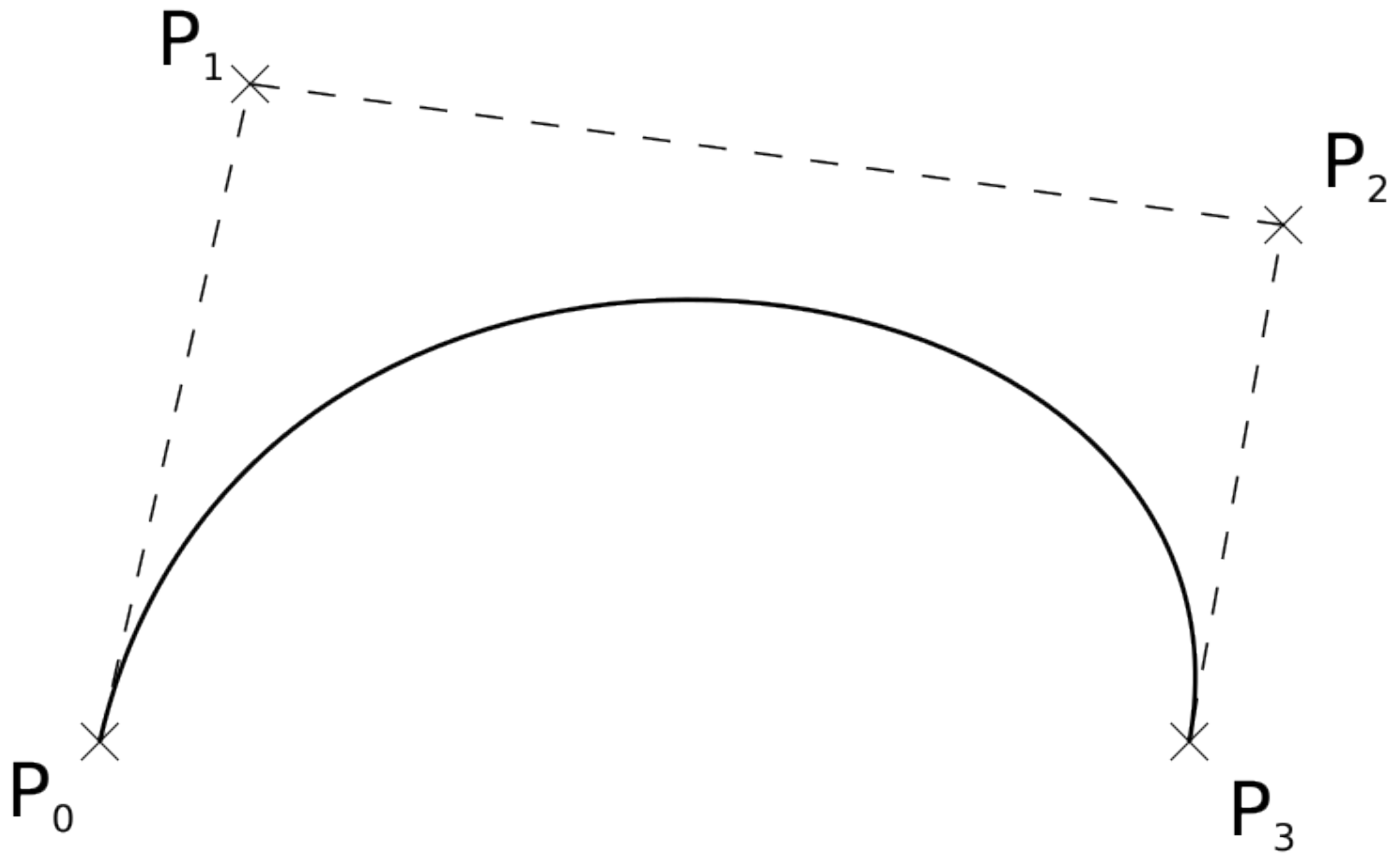
$$\binom{m}{i} = \frac{m!}{(m-i)!\, i!}$$

$$\begin{aligned}
\sum_{i=0}^{m} B_i^m(t) &= \sum_{i=0}^{m} \binom{m}{i} t^i (1-t)^{m-i} \\
&= \sum_{i=0}^{m} \binom{m}{i} s^{m-i} t^i \qquad \text{let } s = 1 - t \\
&= (s+t)^m \\
&= ((1-t)+t)^m \\
&= 1^m = 1
\end{aligned}$$

Bezier Curves

Properties

- Endpoints P_0, P_n are interpolated
- Curve lies within the convex hull of the control polygon
- Curve is affine invariant
- Curve is tangential to the control polygon at the endpoints
- Curve has a fast evaluation algorithm



Wikipedia