CPSC 424 - Tutorial 3

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Degree Elevation

Representing the same Bezier curve, but with a higher degree polynomial.

But why is it useful?

1. Vector format and font type standards. E.g. PostScript, OpenType
2. More degrees of freedom
3. Easy to join Bezier curves for a Bezier surface

Degree Elevation

Given the existing $m+1$ points $b_0 \ldots b_m$, the formula for the new $m+2$ points is:

$$\mathrm{p}_i = \frac{1}{m+1}(i \cdot \mathrm{b}_{i-1} + (m+1-i)\mathrm{b}_i)$$

$$p_0 = b_0, p_{m+1} = b_m$$

Degree Elevation

Exercise:

$$\mathrm{p}_i = \frac{1}{m+1}(i \cdot \mathrm{b}_{i-1} + (m+1-i)\mathrm{b}_i)$$

$$p_0 = b_0, p_{m+1} = b_m$$

Given $b_0 = (0,0), b_1 = (2,4), b_2 = (3,1)$, use the degree elevation formula to find the four control points to specify the equivalent curve.

Try it out here: https://www.desmos.com/calculator/s6t8tmrheu
Credit: Jerry Yin (one of our awesome past TAs!)

Degree Elevation

Given $b_0 = (0,0), b_1 = (2,4), b_2 = (3,1)$, use the degree elevation formula to find the four control points to specify the equivalent curve.

$$p_0 = b_0 = (0,0)$$

$$p_1 = \frac{1}{3}((0,0) + (2)(2,4)) = (\frac{4}{3}, \frac{8}{3})$$

$$p_2 = \frac{1}{3}((2)(2,4) + (1)(3,1)) = (\frac{7}{3}, 3)$$

$$p_3 = b_2 = (3,1)$$

Degree Elevation

Exercise:

Does running degree elevation until you have $2m + 1$ control points give you the same control points as running one iteration of the Bézier subdivision algorithm?

Degree Elevation

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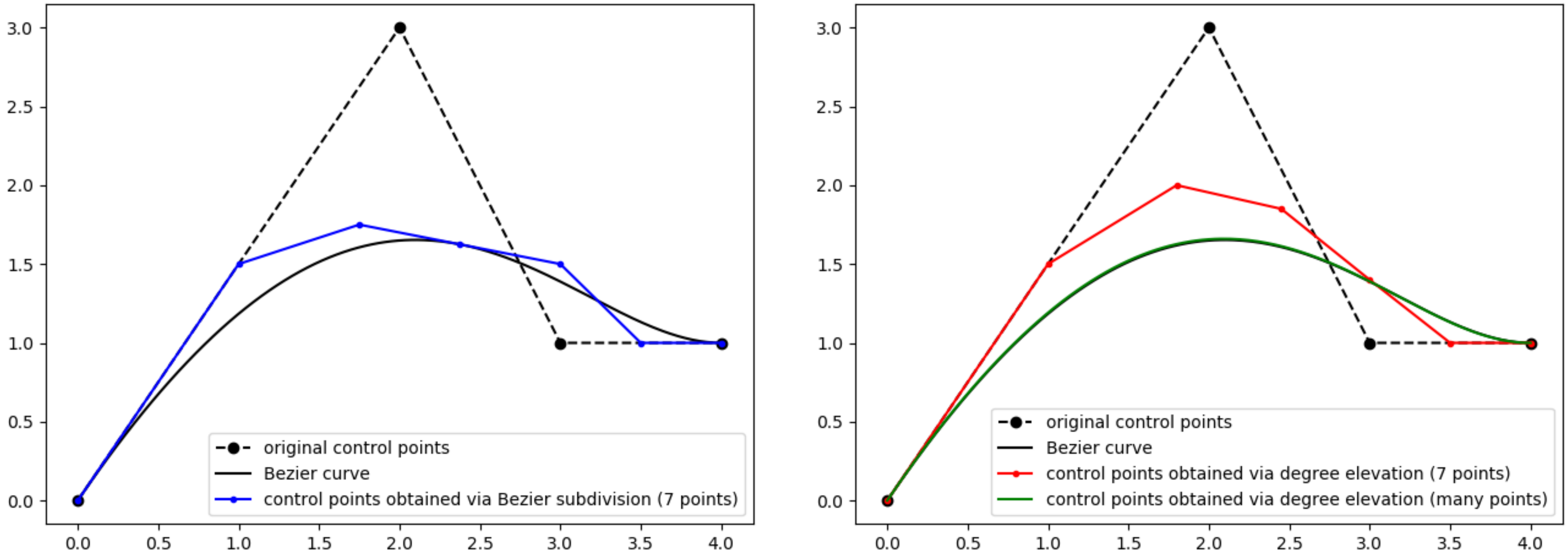
Does running degree elevation until you have $2m + 1$ control points give you the same control points as running one iteration of the Bézier subdivision algorithm?

No, the control points will be different. Recall that Bézier subdivision will give you a set of control polygons, whereas degree elevation gives you a single control polygon. The shared points between the control polygons from the Bézier subdivision lie on the limit curve, and simply combining all control polygons to make a single control polygon will give a different curve entirely.

Degree Elevation



Continuity

We (partially) formalized what we mean for a function to be smooth. All continuous functions belong in a parametric continuity class ($C_0, C_1,$ etc.)

A function is C_0 continuous iff there are no jumps in its value.

A function is C_k continuous iff the first $1...k$th derivatives exist and are continuous (no jumps in its value).

Note that if a function is C_k continuous, then it is also C_j continuous for $j < k$.

Continuity

In addition to parametric continuity, we have a notion of geometric continuity (G_0, G_1, etc.).

A function is G_0 continuous iff there are no jumps in its value (same as C_0)

A function is G_1 continuous if the direction (but not necessarily the magnitude) of its tangent is continuous, and G_k continuous follows the same.

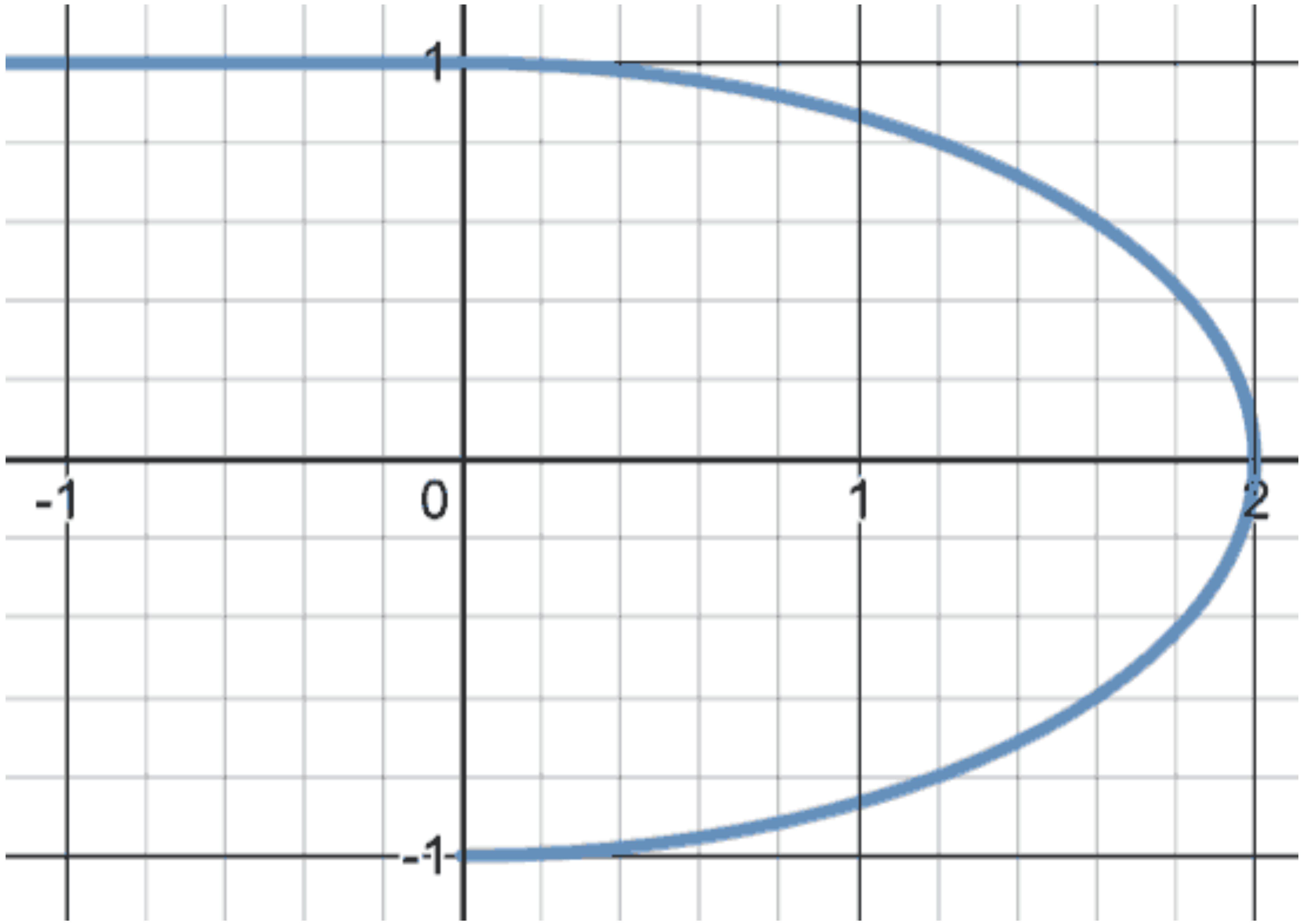
Note that C_k implies G_k continuity.

Continuity

For example, this function here:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{cases} \begin{bmatrix} t \\ 1 \end{bmatrix} & t < 0 \\ \begin{bmatrix} 2\sin t \\ \cos t \end{bmatrix} & 0 \leq t \leq 1 \end{cases}$$

Is G_1 but not C_1 continuous because its derivative at $t = 0$ from the left is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ but its derivative from the right is $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$



Continuity

Exercise 1.

Determine the continuity (C_1, C_2, etc.) of the function:

$$f(x) = \begin{cases} (x+1)^3/3 & 0 \le x < 1 \\ x^2 + 2x - 1/3 & 1 \le x \le 2. \end{cases}$$

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$$\lim_{x \to 1^-} f(x) = \frac{(1+1)^3}{3} = \frac{8}{3},$$

$$\lim_{x \to 1^+} f(x) = 1^2 + 2(1) - \frac{1}{3} = \frac{8}{3},$$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) \implies$ Continuous at $x = 1$ (C^0) .

Continuity

Exercise 1.

Determine the continuity (C_1, C_2, etc.) of the function:

$$f(x) = \begin{cases} (x+1)^3/3 & 0 \le x < 1 \\ x^2 + 2x - 1/3 & 1 \le x \le 2. \end{cases}$$

$$f'(x) = \begin{cases} (x+1)^2 & 0 \le x < 1, \\ 2x + 2 & 1 \le x \le 2. \end{cases}$$

$$\lim_{x \to 1^-} f'(x) = (1+1)^2 = 4,$$

$$\lim_{x \to 1^+} f'(x) = 2(1) + 2 = 4 \implies \text{Differentiable at } x = 1 \ (C^1)\,.$$

Continuity

Exercise 1.
Determine the continuity (C_1, C_2, etc.) of the function:

$$f(x) = \begin{cases} (x+1)^3/3 & 0 \le x < 1 \\ x^2 + 2x - 1/3 & 1 \le x \le 2. \end{cases}$$

$$f''(x) = \begin{cases} 2(x+1) & 0 \le x < 1, \\ 2 & 1 \le x \le 2. \end{cases}$$

$$\lim_{x \to 1^-} f''(x) = 2(1+1) = 4,$$

$\displaystyle\lim_{x \to 1^+} f''(x) = 2 \implies$ Not twice differentiable at $x = 1$ ($\cancel{C^2}$) .

Continuity

Exercise 2.
Determine the parametric continuity (C_1, C_2, etc.) and geometric continuity (G_1, G_2) of each of the curves:

$$f_1(t) = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{cases} \begin{bmatrix} (t+1)^3/3 \\ t^3 \end{bmatrix} & 0 \le t < 1 \\ \begin{bmatrix} t^2 + 2t - 1/3 \\ t^3/2 + 3t/2 - 1 \end{bmatrix} & 1 \le t \le 2 \end{cases}$$

$$f_2(t) = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{cases} \begin{bmatrix} t^3/6 + t^2/2 + t/2 - 1/6 \\ t^2 \end{bmatrix} & 0 \le t < 1 \\ \begin{bmatrix} t^2 \\ t^2 \end{bmatrix} & 1 \le t < 2 \\ \begin{bmatrix} 3t^2/2 + t - 4 \\ t^2 \end{bmatrix} & 2 \le t \end{cases}$$

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$$f_1(t) = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{cases} \begin{bmatrix} (t+1)^3/3 \\ t^3 \end{bmatrix} & 0 \le t < 1 \\ \begin{bmatrix} t^2 + 2t - 1/3 \\ t^3/2 + 3t/2 - 1 \end{bmatrix} & 1 \le t \le 2 \end{cases}$$

$$\lim_{t \to 1^-} f_1(t) = \begin{bmatrix} \frac{(1+1)^3}{3} \\ 1^3 \end{bmatrix} = \begin{bmatrix} \frac{8}{3} \\ 1 \end{bmatrix},$$

$$\lim_{t \to 1^+} f_1(t) = \begin{bmatrix} 1^2 + 2(1) - \frac{1}{3} \\ \frac{1^3}{2} + \frac{3(1)}{2} - 1 \end{bmatrix} = \begin{bmatrix} \frac{8}{3} \\ 1 \end{bmatrix} \implies C^0, G^0.$$

Continuity

Exercise 2.
Determine the parametric continuity (C_1, C_2, etc.) and geometric continuity (G_1, G_2) of each of the curves:

$$f_1(t) = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{cases} \begin{bmatrix} (t+1)^3/3 \\ t^3 \end{bmatrix} & 0 \le t < 1 \\ \begin{bmatrix} t^2 + 2t - 1/3 \\ t^3/2 + 3t/2 - 1 \end{bmatrix} & 1 \le t \le 2 \end{cases}$$

$$f_1'(t) = \begin{cases} \begin{bmatrix} (t+1)^2 \\ 3t^2 \end{bmatrix} & 0 \le t < 1, \\ \begin{bmatrix} 2t+2 \\ \frac{3t^2}{2} + \frac{3}{2} \end{bmatrix} & 1 \le t \le 2. \end{cases}$$

$$\lim_{t \to 1^-} f_1'(t) = \begin{bmatrix} (1+1)^2 \\ 3(1)^2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix},$$

$$\lim_{t \to 1^+} f_1'(t) = \begin{bmatrix} 2(1) + 2 \\ \frac{3(1)^2}{2} + \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \implies C^1, G^1.$$

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$$f_1(t) = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{cases} \begin{bmatrix} (t+1)^3/3 \\ t^3 \end{bmatrix} & 0 \le t < 1 \\ \begin{bmatrix} t^2 + 2t - 1/3 \\ t^3/2 + 3t/2 - 1 \end{bmatrix} & 1 \le t \le 2 \end{cases}$$

$$f_1''(t) = \begin{cases} \begin{bmatrix} 2(t+1) \\ 6t \end{bmatrix} & 0 \le t < 1, \\ [10pt] \begin{bmatrix} 2 \\ 3t \end{bmatrix} & 1 \le t \le 2. \end{cases}$$

$$\lim_{t \to 1^-} f_1''(t) = \begin{bmatrix} 2(1+1) \\ 6(1) \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix},$$

$$\lim_{t \to 1^+} f_1''(t) = \begin{bmatrix} 2 \\ 3(1) \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \implies \cancel{C^2}, \text{ but } G^2.$$

Continuity

Exercise 2.
Determine the parametric continuity (C_1, C_2, etc.) and geometric continuity (G_1, G_2) of each of the curves:

$$f_2(t) = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{cases} \begin{bmatrix} t^3/6 + t^2/2 + t/2 - 1/6 \\ t^2 \end{bmatrix} & 0 \le t < 1 \\ \begin{bmatrix} t^2 \\ t^2 \end{bmatrix} & 1 \le t < 2 \\ \begin{bmatrix} 3t^2/2 + t - 4 \\ t^2 \end{bmatrix} & 2 \le t \end{cases}$$

$$\lim_{t \to 1^-} f_2(t) = \begin{bmatrix} \frac{1}{6} + \frac{1}{2} + \frac{1}{2} - \frac{1}{6} \\ 1^2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$\lim_{t \to 1^+} f_2(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$\lim_{t \to 2^-} f_2(t) = \begin{bmatrix} 2^2 \\ 2^2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix},$$

$$\lim_{t \to 2^+} f_2(t) = \begin{bmatrix} \frac{3(2^2)}{2} + 2 - 4 \\ 2^2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \implies C^0, G^0.$$

Continuity

Exercise 2.

Determine the parametric continuity (C_1, C_2, etc.) and geometric continuity (G_1, G_2) of each of the curves:

$$f_2'(t) = \begin{cases} \begin{bmatrix} \frac{3t^2}{6} + t + \frac{1}{2} \\ 2t \end{bmatrix} & 0 \leq t < 1, \\ \begin{bmatrix} 2t \\ 2t \end{bmatrix} & 1 \leq t < 2, \\ \begin{bmatrix} 3t + 1 \\ 2t \end{bmatrix} & 2 \leq t\,. \end{cases}$$

$$\lim_{t \to 1^-} f_2'(t) = \begin{bmatrix} \frac{3(1)^2}{6} + 1 + \frac{1}{2} \\ 2(1) \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix},$$

$$\lim_{t \to 1^+} f_2'(t) = \begin{bmatrix} 2(1) \\ 2(1) \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix},$$

$$\lim_{t \to 2^-} f_2'(t) = \begin{bmatrix} 2(2) \\ 2(2) \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix},$$

$$\lim_{t \to 2^+} f_2'(t) = \begin{bmatrix} 3(2) + 1 \\ 2(2) \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix} \implies \cancel{C^1, G^1}.$$