CPSC 424 - Tutorial 5

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Parametric Arclength

Given a curve F(t), find the length of the curve for t in range [a, b]:

$$\int_a^b ||F'(t)||\mathrm{d}t = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\mathrm{d}t$$

Intuitively, tangent vector length at each point is related to local length.

Sum (integrate) tangent vector lengths scaled by parameter element dt.

Parametric Arclength

For the following curve, compute the arclengths for the corresponding range:

$$F(t) = \begin{bmatrix} t^3 - 3t \\ 3t^2 \end{bmatrix}, \text{ for } t \in [0, 2]$$

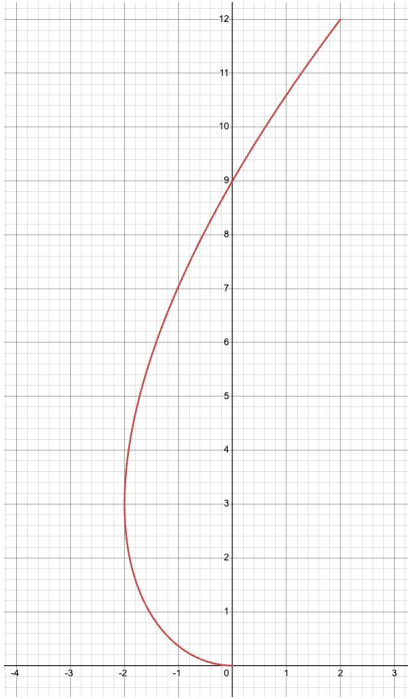
$$\int_a^b ||F'(t)|| \mathrm{d}t = \int_a^b \sqrt{x'(t)^2 + y'(t)^2} \mathrm{d}t$$

Parametric Arclength

$$F(t) = \begin{bmatrix} t^3 - 3t \\ 3t^2 \end{bmatrix}, \text{ for } t \in [0, 2]$$

$$F'(t) = \begin{bmatrix} 3t^2 - 3 \\ 6t \end{bmatrix}$$

$$\begin{aligned}
\int_0^2 ||F'(t)|| \mathrm{d}t &= \int_0^2 \sqrt{(3t^2-3)^2 + (6t)^2} \mathrm{d}t \\
&= \int_0^2 \sqrt{9t^4 + 18t^2 + 9} \mathrm{d}t \\
&= 3 \int_0^2 (t^2 + 1) \mathrm{d}t \\
&= 3(\frac{1}{3}t^3 + t)\Big|_0^2 \\
&= 14
\end{aligned}$$



Re-parameterization

If 2 curves are equivalent in their respective ranges, then there must be some change of parameters ϕ(t) which makes them mathematically equivalent

$$F(\phi(t)) = G(t)$$

Deriving ϕ(t) is not always simple or closed-form. Use known properties of functions as hints

Re-parameterization

For the following 2 curves, find a re-parameterization ϕ(t) which makes them equivalent. If such a re-parameterization exists, prove that your chosen ϕ(t) is valid.

$$F(t) = \begin{bmatrix} \cos^2(t) \\ \sin^2(t)\cos^2(t) \end{bmatrix}, \text{ for } t \in [0, \frac{\pi}{2}]$$

$$G(s) = \begin{bmatrix} 3s+1 \\ -9s^2 - 3s \end{bmatrix}, \text{ for } s \in [-\frac{1}{3}, 0]$$

Hint: Use the following identity $\cos^2 x + \sin^2 x = 1$

$$F(t) = \begin{bmatrix} \cos^2(t) \\ \sin^2(t)\cos^2(t) \end{bmatrix}, \text{ for } t \in [0, \frac{\pi}{2}]$$

$$G(s) = \begin{bmatrix} 3s+1 \\ -9s^2 - 3s \end{bmatrix}, \text{ for } s \in [-\frac{1}{3}, 0]$$

Re-parameterization

$$\text{Let } s = \phi(t) = \frac{1}{3}(\cos^2(t) - 1)$$

$$\begin{aligned} G(s) = \begin{bmatrix} 3s+1 \\ -9s^2 - 3s \end{bmatrix} &= \begin{bmatrix} 3s+1 \\ -3s(3s+1) \end{bmatrix} \\ &= \begin{bmatrix} 3(\frac{1}{3}(cos^2(t) - 1)) + 1 \\ -3(\frac{1}{3}(cos^2(t) - 1))(3s+1) \end{bmatrix} \\ &= \begin{bmatrix} cos^2(t) \\ (1 - cos^2(t))(\cos^2(t) - 1 + 1) \end{bmatrix} \\ &= \begin{bmatrix} cos^2(t) \\ \sin^2(t)\cos^2(t) \end{bmatrix} \\ &= F(t) \end{aligned}$$

