Reduced Coordinates

- Split up model into tiny particles - just like we did for rigid bodies
  - Mass $m_i$, position $x_i$, velocity $v_i$, etc.
- If there are N particles, this is 3N DOF
- Assume constraint equations reduce this to just n DOF
- Figure out n independent parameters $q_1, q_2, \ldots, q_n$ for the particle positions
  - E.g. joint angles
- The $q$'s are the reduced coordinates
- What are the equations of motion for the $q$'s?

Virtual Work

- Start with $F_i = m_i a_i$ for every particle
- Consider the “virtual work” done by a “virtual displacement”
  \[ \sum_{i=1}^{N} F_i \cdot d\bar{x} = \sum_{i=1}^{N} m_i \ddot{x}_i \cdot d\bar{x} \]
- Look at what happens when you change just one of the $q$'s, say $q_j$: \[ d\bar{x}_j = \frac{\partial \ddot{x}_j}{\partial q_j} dq_j \]
  \[ \sum_{i=1}^{N} F_i \cdot \frac{\partial \ddot{x}_i}{\partial q_j} = \sum_{i=1}^{N} m_i \ddot{x}_i \cdot \frac{\partial \ddot{x}_j}{\partial q_j} \]

Introducing Kinetic Energy

\[ \sum_{i=1}^{N} m_i \ddot{x}_i \cdot \frac{\partial \ddot{x}_i}{\partial q_j} = \sum_{i=1}^{N} m_i \left( \frac{d}{dt} \left( \ddot{x}_i \cdot \frac{\partial x_i}{\partial q_j} \right) - \ddot{x}_i \cdot \frac{d}{dt} \left( \frac{\partial x_i}{\partial q_j} \right) \right) \]
\[ = \sum_{i=1}^{N} m_i \left( \frac{d}{dt} \left( \ddot{x}_i \cdot \frac{\partial x_i}{\partial q_j} \right) - \ddot{x}_i \cdot \frac{d}{dt} \left( \frac{\partial x_i}{\partial q_j} \right) \right) \]
\[ = \sum_{i=1}^{N} m_i \left( \frac{d}{dt} \left( \frac{1}{2} \frac{\partial}{\partial q_j} |v_i|^2 \right) - \frac{1}{2} \frac{\partial}{\partial q_j} |v_i|^2 \right) \]
\[ = \frac{d}{dt} \left( \frac{\partial}{\partial q_j} \sum_{i=1}^{N} \frac{1}{2} m_i |v_i|^2 \right) - \frac{\partial}{\partial q_j} \sum_{i=1}^{N} \frac{1}{2} m_i |v_i|^2 \]
\[ = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} \]

Lagrangian Equations of Motion

- Let $f_j = \sum_{i=1}^{N} F_i \cdot \frac{\partial x_i}{\partial q_j}$
  - Called the j'th “generalized force”
- Then the previous equation is
  \[ f_j = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} \]
- With $j=1, 2, \ldots, n$, these are the Lagrangian equations of motion
  - If you work it out, it’s a 2nd order system
General approach

- Figure out what the reduced coordinates are, how to express positions and velocities in terms of the q’s
- Figure out the kinetic energy $T$ as a function of $q$ and $dq/dt$
- For any particular real force, compute the corresponding generalized forces in terms of $q$ and $dq/dt$
- Write down Lagrangian equations of motion
  - Get a system of the form
    \[ C(q,\dot{q})\ddot{q} + b(q,\dot{q}) = f(q,\dot{q}) \]

Implementation

- For any kind of reasonably interesting articulated figure, expressions are truly horrific to work out by hand
- Use computer: symbolic computing
- Input a description of the figure
- Program outputs code that can evaluate terms of differential equation
- Use whatever numerical solver you want (e.g. Runge-Kutta)
- Need to invert $C$ matrix every time step in a numerical
  - Gimbal lock pops up again...