Notes

- Applied math reference has been updated
  - Some new stuff on derivatives, quaternions, and Forward Euler vs. Symplectic Euler

Inertia Tensor Simplified

- Reduce expense of calculating $I(t)$:
  \[
  I(t) = \sum_i m_i (x_i - X)^T (x_i - X)^* \\
  = \sum_i m_i \left[(x_i - X)^T (x_i - X)\delta - (x_i - X)(x_i - X)^T\right]
  \]
- Now use $x_i - X = R p_i$ and use $R^T R = \delta$
  \[
  I(t) = \sum_i m_i \left[p_i^T R^T R p_i \delta - R p_i R^T \right] \\
  = R \left( \sum_i m_i \left[p_i^T (p_i \delta - p_i p_i^T)\right] \right)^T \\
  = R \left( \sum_i m_i \left[p_i^T \delta - p_i p_i^T\right] \right)^T \\
  \]

Inertia Tensor Simplified 2

- So just compute inertia tensor once, for object space configuration
- Then $I(t) = RI_{\text{body}} R^T$
- And $I(t)^{-1} = R(I_{\text{body}})^{-1} R^T$
  - So precompute inverse too
- In fact, since $I$ is symmetric, know we have an orthogonal eigenbasis $Q$
- Rotate object-space orientation by $Q$
  - Then $I_{\text{body}}$ is just diagonal!

Degenerate Inertia Tensors

- Inertia tensor can always be inverted unless all the points of the object line up (object is a rod)
  - Or there’s only one point
- We don’t care though, since we can’t track rotation around that axis anyways
  - So diagonalize $I$, and only invert nonzero elements
Taking the limit

- Letting our decomposition of the object into point masses go to infinity:
  - Instead of sum over particles, integral over object volume
  - Instead of particle mass, density at that point in space

\[ \sum_i m_i \text{foo}(x_i) \rightarrow \iiint \rho(x) \text{foo}(x) \, dx \]

- No big deal
  - In fact, to numerical approximate the integrals, you’ll maybe switch back to the discrete sum over particles...

Approximating Inertia Tensors

- For complicated geometry, don’t really need exact answer
- Could just take the inertia tensor from a simpler geometric figure (will anyone notice?)
- Or numerically approximate integral
  - If we can afford to spend a lot of time precomputing, life is simple
  - Grid approach: sample density...
  - Monte Carlo approach: random samples

Computing Inertia Tensors

- Do the integrals: \( I_{\text{body}} = \iiint \rho(p^T p \delta - pp^T) \, dp \)
- Lots of “fun”
- You may just want to look them up instead
  - E.g. Eric Weisstein’s World of Science on the web
- If not..., align axis perpendicular to planes of symmetry (of \( \rho \)) in object space
  - Guarantees some off-diagonal zeros
- Example: sphere, uniform density, radius R

\[
\begin{pmatrix}
\frac{2}{5} MR^2 & 0 & 0 \\
0 & \frac{2}{5} MR^2 & 0 \\
0 & 0 & \frac{2}{5} MR^2
\end{pmatrix}
\]

Combining Objects

- What if object is union of two simpler objects?
- Integrals are additive
  - But DO NOT USE \( I_1(t) + I_2(t) \):
    - World-space formulas \((x-X)\) use the \( X \) for the object: \( X_1 \) and \( X_2 \) may be different
    - Simplified \( I_{\text{body}} \) formula based on having centre of mass at origin
  - Let’s work it out from the integral of I(t)
- Combined mass: \( M = M_1 + M_2 \)
- Centre of mass of combined object:

\[
X = \frac{\int_{\Omega_1 \cup \Omega_2} \rho x \, \rho \, dx}{\int_{\Omega_1 \cup \Omega_2} \rho} = \frac{M_1 X_1 + M_2 X_2}{M}
\]
Combined Inertia Tensor

\[ I(t) = \int_{\Omega_1 \cup \Omega_2} \rho(x - X)^T (x - X)^* \]
\[ = \int_{\Omega_1} \rho(x - X_1 + X_i - X)^T (x - X_1 + X_i - X)^* + \int_{\Omega_2} \ldots \]
\[ = \int_{\Omega_1} \rho(x - X_i)^T (x - X_i)^* + \int_{\Omega_1} \rho(X_1 - X)^T (x - X_i)^* \]
\[ + \int_{\Omega_2} \rho(x - X_i)^T (X_1 - X)^* + \int_{\Omega_2} \rho(X_1 - X)^T (X_1 - X)^* + \int_{\Omega_2} \ldots \]
\[ = I_1(t) + (X_1 - X)^T \underbrace{\int_{\Omega_1} \rho(x - X_i)^*\rho(x - X_i)}_{0} + \int_{\Omega_1} \rho(x - X_i)^*\rho(X_1 - X)^* \]
\[ + M_1(X_1 - X)^T (X_1 - X)^* + \int_{\Omega_2} \ldots \]
\[ = I_1(t) + M_1(X_1 - X)^T (X_1 - X)^* + I_2(t) + M_2(X_2 - X)^T (X_2 - X)^* \]

Numerical Method

- For advancing V and X, can use any regular scheme (e.g. Symplectic Euler) before
- For advancing angular stuff:
  - Constraint on R makes life a little more interesting
  - R has to stay orthogonal, but regular schemes won’t guarantee that